A Mathematical Model of the Growth and Impact of the United States National Debt

***Abstract*—This paper attempts to provide a simplified model of the formation of national debt with respect to various factors including tax revenue and federal expenditures, as well a mathematical exploration of the relationships between various national economic quantities, in order to realize the impact of various changes to national debt (or its factors) on the economy in general. Our assumptions prior to the modelling process include a Keynesian economy as the economic system of consideration, tax revenue as the sole source of governmental income, and linear dependence of the rate of change of national debt on the difference between federal income and expenditures. We begin by treating the national debt as linearly dependent in rate of change to federal income and federal expenditures, and formulate a differential equation for this relationship by a combination of analytic deriviation and empirical modelling of historical data. We then proceed to obtaining existing mathematical for various economic variables, including consumer expenditures, disposable income, gross domestic product, and net exports, and manipulate and relate these to our original debt formation model to derive conclusions. We then explore the sensitivity of the our debt model to its parameters of time, tax revenue, and federal expenditures.**

***Index Terms*—National Debt, Mathematical Model, Investment, Expenditures**

I. Introduction

THE federal debt of the United States of America is of growing importance to economists analyzing the functioning of the national economy. As such, an understanding of trends in formation and effects of the phenomenon are vital to informed formulation of government economic policies. We attempt a model of the growth of the national debt of the United States, and mathematically analyze the impact of it and its various components on the health of the economy in general. We begin by exploring previous literature on this subject of modelling the various facets of the debt, and after formulating the models, evaluate two economic policies for future years based on our model. We also discuss the sensitivity of our model to various parameters and discuss any descrepencies with literature.

II. Literature Review

After extensive reading of several literature pieces, it can be concluded that there has been a growing and exceptional variety of models evaluating and predicting national debt throughout our history.

An examination of the role of public debt in financial development contrasted the "safe asset" view to the "lazy asset" view. The "safe asset" view is known that bank sectors who loan primarily to the public sector are more likely to lag in developing because the banks would be inefficient although profitable. The examination found that the "safe asset" view takes part in moderate levels of banks' public debt but bank-level regressions stem mainly from the "lazy asset" view [9].

Another study on public debt and tax exempt bonds stated that it would essentially create a "Ponzi type borrowing scheme" where the current young and a limited number of future generations would benefit greatly from this "no public debt" policy. However, after a certain number of years, the welfare would be significantly lower than being without public debt [4]. Also, it is sometimes mistaken that budget deficits are the reasons for a large amount of debt growth when in actuality, stock-flow reconciliation is one of the primary factors in debt dynamics [6].

In a source of literature, it discussed the relationship between capital accumulation and employment by first defining the potential social average investment productivity and then, exploring the effects and rate of investment growth. After analysis, it was found that as soon as investment comes in, growth cannot be left out because investment would contribute to more capital and more labor [10].

On note of short term debt, after confirming their model with Rodrik and Velasco, it was determined that determinants of short term bank lending are not significantly distinguishable between developed and developing countries. [7]

An interesting study was made in developing a neoclassical growth model where it served two purposes: "examining a long-run competitive equilibrium in a growth model" and the effects of the model on government debt. The model indicated that external debt from taxes have two effects; the taxes directly reduce lifetime consumption a taxpayer and reduce the taxpayer's savings as well as capital stock. Internal debt has these effects as well but a further reduction in savings [29].

In studying foreign debt, it was determined that Argentina has a similar model as our nation in studying the growth and stability in foreign debt to output ratio [18].

III. Origin of Data

Our data for gross domestic product, total federal expenditures, and national debt is obtained from historical tables. We obtained the tax revenue data from similar sources, but on four year intervals, and hence had to obtain unavailable years by interpolation - our process for this is summarized in Appendix C.

IV. Assumptions and Notation

The following assumptions hold in the formulation of our model:

1. We assume a Keynesian economy.
2. The only source of governmental income is tax revenue
3. Linear dependancy exists between the rate of change of national debt and the difference between federal income and expenditures
4. The government will always borrow if its federal expenditures are greater than federal income
5. Other assumptions are listed as needed during the modelling process

The following is a listing of nomenclature:

t Time
G(t) Gross Domestic Product
N(t) National Debt
T(t) Tax Revenue
F(t) Federal Expeditures
k Proportionality Constant
E(t) Aggregate Expenditures
C(t) Consumption Expenditures
I(t) Investment Expenditures
M(t) Imports
X(t) Exports
Nx(t) Net Exports
Y(t) Disposable Income
MPC Marginal Propensity to Consume

V. Process Of Modelling

A. Model of Growth of National Debt

We begin our model by asserting the idea that the rate of change of national debt is proportional to the difference between federal income and federal expenditures. Assuming that federal income is equal to the tax revenue of the government, and that the rate of debt change is linearly proportional to this difference between income and expenditure, we can conclude that

$$\frac{dN}{dt} = -k(T(t) - F(t))$$

The above relationship basically summarizes the idea that the rate of change of national debt is inversely proportional to the difference between tax revenue (representational of federal income) and federal expenditures. We now need to derive functions for the tax revenue $T(t)$ and federal expenditures $F(t)$. We begin by extracting data for these two quantities from the sources discussed in the previous section - our plots for these can be obtained in Appendix B.

Our first attempt to deriving these functions is to empirically model historical data using exponential regression by the function $y = ab^x$. The following are the equations we obtained for $T(t)$ and $F(t)$ respectively:

$$\begin{aligned} T(t) &= (7.913257860654 \times 10^{-63}) \cdot (1.078205523621^t) \\ F(t) &= (3.992181705078 \times 10^{-66}) \cdot (1.0826037299682^t) \end{aligned}$$

The R^2 for these fits respectively are 0.86493954564 and 0.987892024693. The equation for rate of change of national debt, by substitution, hence is

$$\begin{aligned} \frac{dN}{dt} &= -k \\ &\cdot ((7.913257860654 \times 10^{-63}) \cdot (1.078205523621^t) \\ &- (3.992181705078 \times 10^{-66}) \cdot (1.0826037299682^t)) \end{aligned}$$

We may now solve the above differential relation by integrating both sides of the equation:

$$\int dN = \int -k(T(t) - F(t))dt$$

$$N = -k \int T(t)dt + k \int F(t)dt$$

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$$\begin{aligned} N &= -k \\ &\cdot ((1.05092387831 \times 10^{-61}) \cdot (1.07820552356^t) \\ &- (5.02990042349 \times 10^{-65}) \cdot (1.08260372997^t)) \end{aligned}$$

We now try to convert our regression functions for the tax revenue and expenditure data into piecewise functions to obtain a better fit. Explanations obtained from several of our references for the breaks in the piecewise can be found in Appendix D and E.

$$F(t) = \begin{cases} 1.62045 \times 10^{-135} \cdot 1.17501^t & 1948 \le t < 1952 \\ 2.95963 \times 10^{-41} \cdot 1.05138^t & 1952 \le t < 1959 \\ 1.00668 \times 10^{-55} \cdot 1.06942^t & 1959 \le t < 1967 \\ 1.61036 \times 10^{-65} \cdot 1.08180^t & 1967 \le t < 1974 \\ 2.88048 \times 10^{-74} \cdot 1.09297^t & 1974 \le t < 1991 \\ 8.86627 \times 10^{-291} \cdot 1.03692^t & 1991 \le t < 1999 \\ 3.25727 \times 10^{-48} \cdot 1.06039^t & t \ge 1999 \end{cases}$$

$$T(t) = \begin{cases} 8.84308 \times 10^{-109} \cdot 1.13847^t & 1948 \le t < 1954 \\ 1.91498 \times 10^{-47} \cdot 1.05888^t & 1954 \le t < 1974 \\ 1.49388 \times 10^{-65} \cdot 1.08164^t & 1974 \le t < 1999 \\ 83.71t^3 - 502283.21t^2 & \\ +1004492971.46t & \\ -669612260899.18 & t \ge 1999 \end{cases}$$

Refer to Appendix F for a plot of these functions into the original data.

B. Model of Impact of National Debt

We begin to explore the impact of the national debt of the United States on its national economic infrastructure by asserting the fact that

$$E(t) = C(t) + I(t) + F(t) + NX(t)$$

where

$$\begin{aligned} C(t) &= a + bY^d(t) \\ I(t) &= I_0 + jY \\ Y^d(t) &= Y - T \\ M(t) &= M_0 + mY \end{aligned}$$

and

$$T(t) = T_0 + \tau Y$$

We can now derive these functions that we obtained from TBA, in order to get relationships between the various rates of change:

$$\begin{aligned} \frac{dC}{dt} &= b\frac{dY}{dt} = -b\tau\frac{dY}{dt} \\ \frac{dI}{dt} &= j\frac{dY}{dt} \\ \frac{dY}{dt} &= -\frac{dT}{dt} = -\tau\frac{dY}{dt} \\ \frac{dT}{dt} &= \tau\frac{dY}{dt} \end{aligned}$$

We also assume a Keynesian economy, where Say's law that demand would always equal supply in an economy does not hold and consumer spending is proportional to income by a propensity of consumption. Since the Gross Domestic Product of a nation can be represented by its income, and consumer spending is equivalent to the aggregate expenditures of an economy, we can assert that

$$E(t) = E(0) + MPC(G(t))$$

We can now formulate an expression for the marginal propensity to consume as

$$MPC = \frac{dE(t)}{dG(t)}$$

We are now interested in the break-even point of the expediture-GDP graph; noting the fact that the break-even point, were all earnings are totally spent, is located at the intersection of a fourty-five degree ray from the origin and the economy's expenditure curve, namely of the following two equations,

$$E(t) = E(0) + MPC(G(t))$$

$$E(t) = G(t)$$

we can quantify the break even point to be

$$B = \frac{E(0)}{1 - MPC} = \frac{E(0)}{1 - \frac{dE(t)}{dG(t)}}$$

On the basis of this idea, we can now redefine our original model of rate of change of national debt to be:

$$\frac{dN}{dt} = \begin{cases} -k(T(t) - F(t)) & G(t) < \frac{E(0)}{1 - \frac{dE(t)}{dG(t)}} \\ k(T(t) - F(t)) & G(t) > \frac{E(0)}{1 - \frac{dE(t)}{dG(t)}} \end{cases}$$

because below the break-even point, the infrastructure of the economy is in the mode of saving, and most likely, paying off its previous debt in order to maintain a balanced budget. Likewise, at a GDP greater than the break-even point, the economy is dissaving and either using off its previous savings to maintain an expenditure below current income, or borrowing externally for that purpose. Note that

$$\frac{dG}{dt} = \left(\frac{1}{1 - MPC}\right) \cdot \frac{dE}{dt}$$

VI. Analysis of Model

We now explore the sensitivity of the model to various parameters within it; we first explore the impact of various increments to tax revenue and federal expenditures on our national debt rate of change equation - the following table list the results of various considered increments:

T(t)	$\frac{dN}{dt}$	$\triangle T$	$\triangle \frac{dN}{dt}$
2000.98	1468.37	-0.900000	0.900000
2001.88	1467.47	-0.090000	0.090000
2001.97	1467.38	-0.009000	0.009000
2001.98	1467.37	-0.000900	0.000900
2001.98	1467.37	-0.000100	0.000100
2001.98	1467.37	0.000000	0.000000
2001.98	1467.37	0.000100	-0.000100
2001.98	1467.37	0.000900	-0.000900
2001.99	1467.36	0.009000	-0.009000
2002.08	1467.27	0.090000	-0.090000
2002.98	1466.37	0.900000	-0.900000

F(t)	$\frac{dN}{dt}$	$\triangle$F	$\triangle\frac{dN}{dt}$
3468.35	1466.37	-0.900000	-0.900000
3469.25	1467.27	-0.090000	-0.090000
3469.34	1467.36	-0.009000	-0.009000
3469.35	1467.37	-0.000900	-0.000900
3469.35	1467.37	-0.000100	-0.000100
3469.35	1467.37	0.000000	0.000000
3469.35	1467.37	0.000100	0.000100
3469.35	1467.37	0.000900	0.000900
3469.36	1467.38	0.009000	0.009000
3469.45	1467.47	0.090000	0.090000
3470.35	1468.37	0.900000	0.900000

From the table, it is clear that $\frac{dN}{dt}$ changes linearly according to k for a particular change in our model's tax revenue or federal expenditure variables. This can be further demonstrated analytically by differentiating our model:

$$\frac{dN}{dt} = -k(T(t) - F(t))$$

$$\frac{dN}{dt} = -kT(t) + kF(t))$$

$$\frac{d^2N}{dt^2} = -k\left(\frac{dT(t)}{dt} - \frac{dF(t)}{dt}\right)$$

One final observation of interest is the fate of debt as time goes on forever; we evaluate this for our model as follows:

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} -k\int T(t)dt + k\int F(t)dt = \infty$$

Clearly, our model predicts that national debt will keep increasing as time progresses, though (TBA) argue otherwise in their cyclic characterization of the trend of debt accumulation and (TBA2) demonstrate this for the economy of Argentina. We could possibly alter our model for this consideration as future work.

VII. Evaluation of Policies Based On Model

We now consider two alternative policies for the time period 2007 to 2018, and evaluate them in terms of impact of national debt, and in that respect, on the economy in general, using our models. Our considered policies are:

1. John McCain's proposed spending increase of \$92.437 billion, along with no increase in payroll tax or in taxes on general public
2. Barack Obama's proposed spending increase of \$292.954 billion, with a positive proposed increase in social security tax and tax cuts for middle class

Consider the tax policies first - both policies propose an expenditure increase, though Policy 1 favor's an increase smaller than that of Policy 2. Reiterating our relationship from the previous section that $\frac{dN}{dt} \propto \frac{1}{\frac{dT}{dt}}$, we can easily see that a larger increase in taxation will in large decrease in national debt. Hence, Policy 1 would result in a slower rate of increase of national debt than Policy 2.

Now, focusing on expenditures, we can reassert our previous relationship that $\frac{dN}{dt} \propto \frac{dF}{dt}$, and hence, a larger expenditure policy would mean a faster growth of the national debt. From this discussion, we can conclude that Policy 1 would result in a slower rate of increase of national debt from the perspective of taxation and expenditure, compared to Policy 2, which will result in a faster one.

VIII. Conclusion

In this paper, we formulated a model for growth and impact of the national debt of the United States of America. Our model for growth was composed of a differential equation modelling the rate of change of national debt linearly with respect to federal income (which was assumed to be primarily composed of taxes) and federal expenditures. We then explored the literature for various existing relations between the economic variables of tax and expenditure, and explored the relationships between them, and in consequence, the national debt and its effect on the economy. Next, we analyzed the sensitivity of various parameters of our model on the prediction of the rate of change of national debt, and then culminated with the application of the model to evaluation of two alternative policies of John McCain and Barack Obama, in terms of taxation and planned expenditure.

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