Aorta Radius

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Abstract

We utilize the models of [1] for the scaling laws of the vascular networks of biological organisms in order to derive a proof for the idea that the radius of an aorta blood vessel of an animal, r_0, scales with its mass, M as $M^{\frac{3}{8}}$. We then extend our logic to the length of an aorta blood vessel, l_0 and demonstrate that it scales with an organism's mass as $M^{\frac{1}{4}}$.

1 Introduction

There has been an immense increase the amount of mathematical modelling of biological processes in the recent decades. One important biological system of interest for modellers is the vascular blood network that services the body of an organism with blood - it has been observed that this branching network has some quite interesting and predictable properties from a modelling perspective, that are also highly similar to technological networks such as the World Wide Web and citation networks such as those generated by online citation tracking websites as CiteSeer. One pioneering work in this area of modelling blood vessel networks is that of [1], which assumes a branching network setup of blood vessels in an organism, and derives several modelling equations for various properties of this network, including the total number of capillaries, total volume of blood, and the like. The same paper also touches upon the scaling laws of quantities such as the total number of capillaries in relation to other things as mass of an organism. Interesting observations include the derivation of the $-\frac{4}{3}$ law common in plant ecology, and a confirmation of the $\frac{3}{4}$ power law derived by [2] for the basal metabolic rate of an organism in relation to its mass. We adopt the ideas of [1] in order to prove two hypothesized scaling relationships - that the radius of an aorta scales with the mass of an organism as $M^{\frac{3}{8}}$, and that the length of the aorta scales with the same mass as $M^{\frac{1}{4}}$.

2 Scaling of Radius

We begin our proof with the equation of [1] for Q_0, the fluid flow rate through the aorta, as

$$Q_0 = N_k Q_k = N_k \pi r_k^2 u_k = N_N \pi r_N^2 u_N$$

hence asserting that

$$N_k \pi r_k^2 u_k = N_N \pi r_N^2 u_N$$

Since N_k represents the number of capillaries at level k, and in our case, $k = 0$ to represent the aorta at the base level, we can assert that $N_0 = 1$, and rewrite the above equation as

$$\pi r_k^2 u_k = N_N \pi r_N^2 u_N$$

Noticing that $u_k$and u_N are mean velocities of fluid flow, and hence constant, we can write the above equation without the constants as the following:

$$r_k^2 = N_N r_N^2$$

We know from [1] that an approximation for N_N is

$$N_N \approx \left(\frac{M}{M_0}\right)^{\frac{3}{4}}$$

Substituting this approximation expression into our previous equation, we obtain

$$r_k^2 = \left(\frac{M}{M_0}\right)^{\frac{3}{4}} r_N^2$$

Making $r_N^2$the subject of the equation,

$$\frac{r_k^2}{r_N^2} = \left(\frac{M}{M_0}\right)^{\frac{3}{4}}$$

and taking the square root of both sides

$$\sqrt{\frac{r_k^2}{r_N^2}} = \sqrt{\left(\frac{M}{M_0}\right)^{\frac{3}{4}}}$$

we obtain

$$\frac{r_k}{r_N} = \left(\frac{M}{M_0}\right)^{\frac{3}{8}}$$

and treating r_N and M_0 as proportionality constants, arrive to the conclusion that the radius of the aorta scales with mass of an animal as $M^{\frac{3}{8}}$.

3 Scaling of Length

We now prove that the length of the aorta blood vessel should scale with mass as $M^{\frac{2}{3}}$. We initiate our proof process by iterating the equation derived by [1] for the total volume of blood serviced by all capillaries of an organism:

$$V = \pi\rho^2 l_N N_N \approx C l_N^3 N_N$$

If we use the logic from the previous section and write

$$C l_k^3 N_k = C l_N^3 N_N$$

we can now use the same method of proof used before; since $k = 0$ in our case,

$$C l_0^3 N_0 = C l_N^3 N_N$$

and realizing that $N_0 = 1$ because it is the number of capillaries at level zero, or the aorta itself, the equation can be formulated as

$$C l_0^3 = C l_N^3 N_N$$

We can now drop the constants, including the l_N which is the total number of capillaries, and assert that

$$l_0^3 = N_N$$

and recalling the $N_N = \left(\frac{M}{M_0}\right)^{\frac{3}{4}}$, say by substitution that

$$l_0^3 = \left(\frac{M}{M_0}\right)^{\frac{3}{4}}$$

We now take the cube root of both sides to obtain

$$l_0 = \left(\frac{M}{M_0}\right)^{\frac{1}{4}}$$

Treating M_0 as an arbitrary constant, we have proven that the length of the aorta, l_0, scales with mass, M, as $M^{\frac{1}{4}}$.

4 Conclusion

We began by an overview of the models of [1] for the vascular blood networks of animals. We then proceeded with hypothesizing that the radius of an aorta scales with the mass of an organism as $M^{\frac{3}{8}}$ and the length of the aorta scales with the same mass as $M^{\frac{1}{4}}$. Then, we set about to successfully prove these two scaling relationships by a manipulation of the models of [1] for mean velocities of fluid flow rate through blood vessels and the total volume of blood serviced by capillaries in an animal.

References

[1] West, Brown, & Enquist (1998). A General Model for the Origin of Allometric Scaling Laws of Biology. *Science*

[2] Kleiber, M (1938). Food value and metabolic unit of body size. Journal of Animal Science - Am Soc Animal Sci