Logarithmic Spiral

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Abstract

One of the routines of a falcon impacted by the angled alignment of its eyes is its trajectory during an attack on a prey. We explore the equiangular path of a falcon during its attack flight, and analytically prove it as a logarithmic spiral of form $r = ae^{b\theta}$.

1 Introduction

Since the deep fovea of a falcon are angled away from the its head axis [1], the bird generally has to maintain its visual acuity as well as aerodynamic efficiency through a constant angular flight of 40 degrees from its alignment with the prey [2]. Refer to Figure 1 for a visual overview of this flight trajectory of a falcon prey hunting. It is experimentally known in [2] that the angular path is of the form of a logarithmic spiral, and we attempt to verify this theory mathematically.

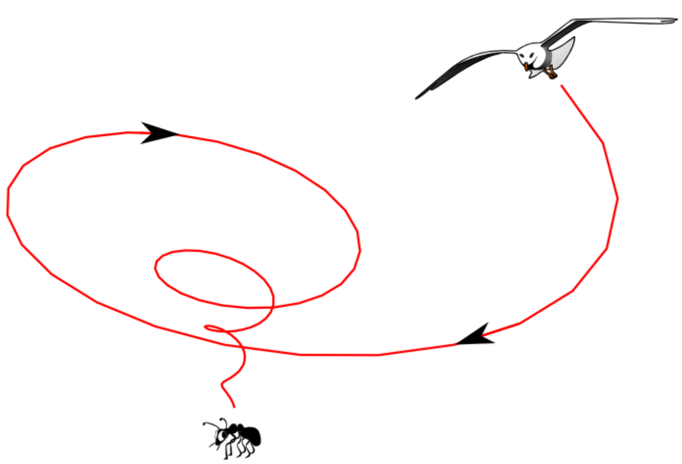

Figure 1: An illustration of the spiral path followed by a falcon during attack on a target

2 Analytical Proof

We now prove the hypothesis that the trajectory of a falcon in $\Re^2$ during an attack is in the form of a logarithmic spiral of the general form $r = ae^{b\theta}$, where r is the radius of the spiral for a given θ, and a and b are arbitrary constants, functions of which are discussed later.

Figure 2 illustrates our initial setup for a small fraction of the equiangular trajectory of the falcon during attack.

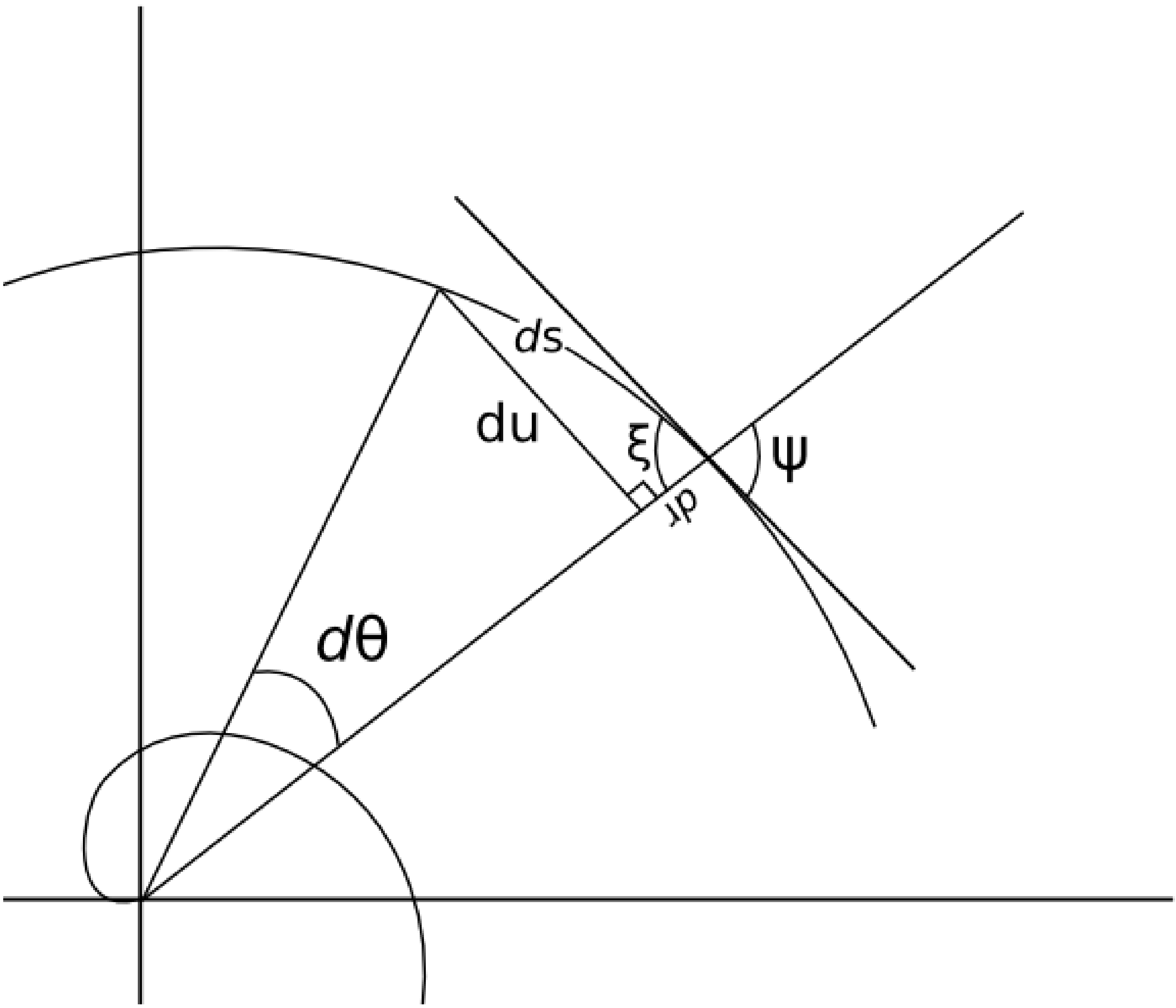

Figure 2: Mathematical setup of a part of an attacking falcon's trajectory with an equiangular requirement

A preliminary step to our proof is to find $\frac{\partial r}{\partial \theta}$ for a general logarithmic spiral of the form

$$r = ae^{b\theta}$$

which can be computed by evaluating the derivative of both sides, yeilding

$$\frac{\partial r}{\partial \theta} = b \cdot ae^{b\theta}$$

in which the second term of the right-hand side can be substituted using the previous equation to obtain

$$\frac{\partial r}{\partial \theta} = br$$

Setting this result for aside later use, we can now proceed with our usual proof process. The first step is to realize that as the term $\partial\theta$ shown in Figure 2 approaches zero, the angle ξ approaches ψ, since the line u equals approximately the tangent line shown at the point. Also, the latter point is important to emphasize - as $\partial\theta$ approaches zero, u approaches ∂s. Knowing these facts, we can now consider the right triangle shown in the figure to say that

$$tan\psi = \frac{\partial u}{dr}$$

is the same as

$$tan\psi = \frac{\partial s}{dr}$$

after which, modifying the latter equation yeilds

$$\frac{1}{tan\psi} = \frac{\partial r}{ds}$$

which is equal to

$$cot\psi = \frac{\partial r}{ds}$$

due to the fact that the cotangent of an angle equal to its inverse tangent. Also, since arc length, s, is

$$s = r\theta$$

we can conclude that by differentiation of both sides,

$$\partial s = r\,\partial\theta$$

for the arc shown in Figure 2. By substitution into the cotagent equation of ψ, we have

$$cot\psi = \frac{\partial r}{rd\theta}$$

which can be rewritten as

$$\frac{1}{r}\partial r = cot\psi\,\partial\theta$$

Integrating both sides, we get

$$\Delta\ \frac{1}{r}\partial r = \Delta\ cot\psi\,\partial\theta$$

Since ψ is constant, we can write the above as

$$\Delta\ \frac{1}{r}\partial r = cot\psi\,\Delta\ \partial\theta$$

which, after the integration process, results in

$$ln\,r = cot\psi \cdot \theta + C$$

Now, recalling the preliminary calculation we did to obtain $\frac{\partial r}{\partial\theta} = br$, and also iterating from the inital part of our main proof that $cot\psi = \frac{\partial r}{rd\theta}$ or $rcot\psi = \frac{\partial r}{d\theta}$, we assert that by substitution that

$$rcot\psi = br$$

or

$$cot\psi = b$$

With this in mind, we can go back to our previous result of integration and by substitution obtain,

$$ln\,r = b\theta + C$$

Raising both sides to the power of e, we get

$$r = e^{b\theta+C}$$

which can also be formulated as

$$r = e^{b\theta} \cdot e^{C}$$

of which the second term can be just be written as C to get

$$r = e^{b\theta} \cdot C$$

Assuming a as an arbitrary constant, the latter equation can be written as

$$r = ae^{b\theta}$$

which is the general equation of a logarithmic spiral. This proves our original hypothesis that the trajectory of the path of a falcon follows a logarithmic spiral of form $r = ae^{b\theta}$.

3 Conclusion

We began by adopting the idea that falcons attack their prey through a route that is of the form of a logarithmic spiral. Assuming a 40 degree constant angle from literature, we move on to prove the idea that the requirement of an equiangular nature of the attack path is what leads to the logarithmic spiral form of the trajectory. We come to this support of our hypothesis by realizing geometrically illustrating a typical equiangular trajectory alike to that of a falcon's, and then manipulating algebraic representations of it until we arrive at the standard equation of a logarithmic spiral, proving our hypothesis.

References

[1] Curved paths in raptor flight: Deterministic models. Journal of Theoretical Biology. Volume 242, Issue 4, 21 October 2006, Pages 880-889

[2] Flying Along a Logarithmic Spiral. http://www.sciencemag.org