Continued Fractions

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Abstract

In this paper, we discuss the nature specific continued fractions. We begin by examining and verifying the fact that the Golden Ratio, Φ, can be expressed as a continued fraction, and then move on to reiterating the same process for other irrational numbers such as $\sqrt{2}$. For the given process of verification, we use both analytical and computational techniques. We conclude by an examination and comparison of convergence rates of the continued fractions at successive truncation points for the various examined irrational numbers, and conclude that the Golden Ratio has the smallest convergence rate.

1 The Golden Ratio as a Continued Fraction

The Golden Ratio is a value in geometry and analysis that has fascinated mathematicians for centuries, and instances of it as a descriptor can be seen in numerous settings of nature [1]. We are given the idea that the Golden Ratio is expressible as a continued fraction [2], and asked to prove it. We begin by a restatement of the format of the continued fraction in context, followed by analytical and computational proofs. The hypothesis is that the Golden Ratio, Φ, can be written as:

$$\Phi = 1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}}}}}$$

To prove this mathematically, we begin with the known fact that the Golden Ratio is a number, x, whose square equals one added to itself, that is

$$x^2 = x + 1$$

We can now perform some manipulation of this to get a part of the continued fraction, that is, by dividing the previous equation by x to get

$$x = 1 + \frac{1}{x}$$

We can now realize that the x in the right-hand side of the equation can be substituted for the whole right-hand side expression as described the the equation, so substituting once, we get

$$x = 1 + \frac{1}{\left(1 + \frac{1}{x}\right)}$$

Obviously, the x in the denominator reappears, and we can continue substitution of it over and over again, an infinite number of time, to obtain Φ

$$x = 1 + \frac{1}{\left(1 + \frac{1}{\left(1 + \frac{1}{\left(1 + \frac{1}{1 + \dots}\right)}\right)}\right)}$$

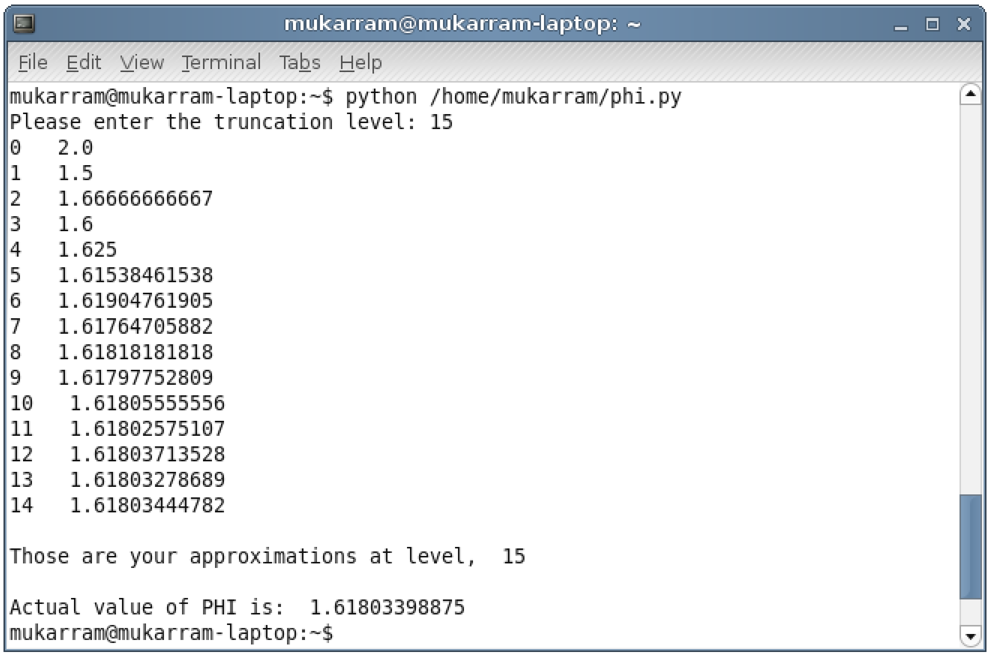
and thus, that leads us to conclude that

$$\Phi = 1 + \frac{1}{\left(1 + \frac{1}{\left(1 + \frac{1}{\left(1 + \frac{1}{1 + \dots}\right)}\right)}\right)}$$

our original hypothesis. Now that we have a mathematical proof of the fact that the Golden Ration can be expressed as a continued fraction such as the one just listed, we can proceed with detailing a computational demonstration of this continued fraction and its ability to approximate Φ. For the purpose of this paper, we wrote a computer program in the Python programming language that prompts the user for the level of truncation of the continued fraction, and then calculates the above fraction to that particular level. The listing of the program is as follows:

```
import math

level = int(raw_input("Please enter the truncation level: "))
ans = 1.0

for i in range(0, level):
      ans = 1.0 + (1.0 / ans)
      print i, ' ', ans

print '' print 'Those are your approximations at level, ', level
print ''

phi = 0.5 * (1 + math.sqrt(5))

print 'Actual value of PHI is: ', phi
```

The given program works as follows: after prompting the user for the truncation level, it starts at the bottom of the resultant continued fraction, and begins evaluation from that point, displaying the value of Φ at each successive level above. The following is an example of the results of a run with the truncation level being set to fifteen:



The results of the program for various other truncation level shall be discussed in Section 3, in the context of comparison with that of other selected irrational numbers.

2 The Square Root of Two as a Continued Fraction

In the previous section, we discussed that the Golden Ratio is expressible as a continued fraction - for the purpose of comparison in the next section, we now prove the continued fraction expressibility of another well known irrational number, the square root of two. Specifically, we want to illustrate that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}}}}}$$

For our purposes, we now begin by first detailing another proof: that given the expression $\sqrt{2} = 1 + \frac{1}{x}$, x satisfies the equation $x^2 = 2x + 1$. So, starting with the first equation and manipulating it to get to the second,

$$\sqrt{2} = 1 + \frac{1}{x}$$

we first multiply both sides by x to get

$$(\sqrt{2})x = x + 1$$

and then squaring both sides returns

$$2x^2 = (x + 1)^2$$

which, after expansion of the binomial in the right-hand side, yields

$$2x^2 = x^2 + 2x + 1$$

which is equivalent to our original hypothesized equation:

$$x^2 = 2x + 1$$

Now that this equation is proved to be true for all x, we can use the same idea as for the Golden Ratio in the first section, by asserting

$$x = 2 + \frac{1}{x}$$

due to the division of both sides by x. The x in the denominator can now be replaced by the right-hand side expression over and over, just as done in the last section, to obtain

$$x = 1 + \frac{1}{\left(2 + \frac{1}{\left(2 + \frac{1}{\left(2 + \frac{1}{2 + \ldots}\right)}\right)}\right)}$$

which leads us to conclude that

$$\sqrt{2} = 1 + \frac{1}{\left(2 + \frac{1}{\left(2 + \frac{1}{\left(2 + \frac{1}{2 + \ldots}\right)}\right)}\right)}$$

As with the Golden Ratio, we also wrote a program to demonstrate the evaluation of the continued fraction for the square root of two, and the listing of the code, quite similar to the one for Φ, is given below:

```
import math

level = int(raw_input("Please enter the truncation level: "))
```

```
ans = 1.0

for i in range(0, level):
      if i == level - 1:
            ans = 1.0 + (1.0 / ans)
            print i, ' ', ans
      else:
            ans = 2.0 + (1.0 / ans)
            print i, ' ', ans

print ''
print 'Those are your approximations at level, ', level print ''

sqrt2 = math.sqrt(2)

print 'Actual value of sqrt(2) is: ', sqrt2
```

3 Rates of Convergence of Continued Fractions

In this section, we compare the rate of convergence of various continued fractions to the actual values that the fractions approximate, by observing the values of the fractions at various truncation levels. The irrational numbers we are going to consider are the Golden Ratio, Φ, the square root of 2, $\sqrt{2}$, the constant e, and π. We are given the following two continued fractions for the latter pair as valid:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cfrac{5}{6 + \cfrac{6}{7 + \ldots}}}}}}}$$

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \cfrac{9^2}{6 + \cfrac{11^2}{6 + \cfrac{13^2}{6 + \ldots}}}}}}}$$

By manipulating the computer programs written for Φ and $\sqrt{2}$, we can get the truncation levels of the two irrational numbers given above. The following table shows the values of the continued fractions, as computed by the four programs, at successive integer values up till thirty (where all four fractions evaluated a value equal to the actual value within computer accuracy):

	Φ	$\sqrt{2}$	e
1	1.5	2.33333333333	8.0
2	1.66666666667	2.42857142857	3.375
3	1.6	2.41176470588	2.59259259259
4	1.625	2.41463414634	1.38571428571
5	1.61538461538	2.41414141414	2.72164948454
6	1.61904761905	2.41422594142	2.71777003484
7	1.61764705882	2.41421143847	2.71834885477
8	1.61818181818	2.41421392678	2.71827411168
9	1.61797752809	2.41421349985	2.71828262219
10	1.61805555556	2.4142135731	2.71828175463
11	1.61802575107	2.41421356053	2.71828183473
12	1.61803713528	2.41421356269	2.71828182797
13	1.61803278689	2.41421356232	2.71828182849
14	1.61803444782	2.41421356238	2.71828182846
15	1.6180338134	2.41421356237	2.71828182846
16	1.61803405573	2.41421356237	2.71828182846
17	1.61803396317	2.41421356237	2.71828182846
18	1.61803399852	2.41421356237	2.71828182846
19	1.61803398502	2.41421356237	2.71828182846
20	1.61803399018	2.41421356237	2.71828182846
21	1.61803398821	2.41421356237	2.71828182846
22	1.61803398896	2.41421356237	2.71828182846
23	1.61803398867	2.41421356237	2.71828182846
24	1.61803398878	2.41421356237	2.71828182846
25	1.61803398874	2.41421356237	2.71828182846
26	1.61803398875	2.41421356237	2.71828182846
27	1.61803398875	2.41421356237	2.71828182846
28	1.61803398875	2.41421356237	2.71828182846
29	1.61803398875	2.41421356237	2.71828182846
30	1.61803398875	2.41421356237	2.71828182846

From the given table, we can observe that all four have relatively different rates of convergence to their 'actual' value within computer degree of accuracy, and that the slowest converging irrational number within our examined group of irrational numbers is the Golden Ratio, Φ.

References

[1] Huntley, H. The Divine Proportion - A Study in Mathematical Beauty. Dover Publications

[2] Tung, K. Topics in Mathematical Modelling. Princeton University Press